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The Kowalewski top in the SUSY quantum mechanics.

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Abstract

The Kowalewski top on Lie algebras $o(4)$, $e(3)$ and $o(3,1)$ is embedded in the SUSY quantum mechanics. In two dimensions we give the new prescription for construction of the pairs of integrable systems by using a standard SUSY algebra. At the proposed scheme the Goryachev-Chapligin top is shown to be a natural partner of the Kowalewski top.

Key words: Kowalewski top, Goryachev-Chapligin top, supersymmetrical quantum mechanics, Darboux transformation.

Supersymmetrical quantum mechanics represents a concise algebraic form of the spectral equivalence between different Hamiltonian quantum systems realized by means of the Darboux transformation [1]. While a theory of the one-dimensional supersymmetrical quantum mechanics is now well developed, this is not so for the multidimensional ones and we still are in phase of case studies. The aim of this note is to show that the Kowalewski top and the Goryachev-Chapligin top are the two-dimensional supersymmetric models in quantum mechanics. Notice, that the construction of isospectral two-dimensional Hamiltonians in supersymmetrical quantum mechanics is tightly connected with another problem,

namely with a search for the second integral of motion for the quantum integrable systems. Moreover, recently the SUSY algebra have been applied to the separation of variables for the two-dimensional hamiltonian systems [1, 6].

We recall here the basic algebraic constructions of the supersymmetrical quantum mechanics. All details can be found in [1] and references therein. The intertwining relations between two Hamiltonians $h^{(1)}$ and $h^{(2)}$ with equivalent spectra are given by

$$\begin{aligned} h^{(1)}q^{+} &= q^{+}h^{(2)}\,, & q^{-}h^{(1)} &= h^{(2)}q^{-}\,,\\ h^{(1)}\Psi_n^{(1)} &= E_n\Psi_n^{(1)}\,, & h^{(2)}\Psi_n^{(2)} &= E_n\Psi_n^{(2)}\,,\\ (q^{+})^{\dagger} &= q^{-}\,, \end{aligned} \tag{1}$$

The concise algebraic form of the spectral equivalence is given by the superalgebra for the partners $h^{(1)}$ and $h^{(2)}$ and off-diagonal supercharges

$$\begin{aligned} &H=\begin{pmatrix} h^{(1)} & 0\\ 0 & h^{(2)}\end{pmatrix}, \qquad Q^{+}=\left(Q^{-}\right)^{\dagger}=\begin{pmatrix} 0 & 0\\ q^{-} & 0\end{pmatrix},\\ &(Q^{+})^2=(Q^{-})^2=0\,, \qquad \left[H,Q^{\pm}\right]=0\,. \end{aligned} \tag{2}$$

Since the super charges $Q^{\pm}$ commute with the Hamiltonian H one expects that the closing of the superalgebra will lead to the symmetry operator R (central charge) [1]

$$\{Q^{+},Q^{-}\}=R\,, \qquad [H,R]=0\,, \qquad R=\begin{pmatrix} R_1 & 0\\ 0 & R_2\end{pmatrix}, \tag{3}$$

where braces $\{,\}$ mean anticommutator of quantum operators. In the two-dimensional case the closing of the SUSY algebra leads to the integrability of the corresponding dynamical system and R is the second integral of motion.

Let us consider quantum (or classical) dynamical systems on algebra $\mathfrak{g}=o(4)$, $e(3)$, $o(3,1)$ with generators obeying the following commutator relations:

$$\begin{aligned} &[l_i\,,l_j\,]=-i\eta\varepsilon_{ijk}\,l_k\,, \qquad [l_i\,,g_j\,]=-i\eta\varepsilon_{ijk}\,g_k\,,\\ &[g_i\,,g_j\,]=i\eta P\varepsilon_{ijk}\,l_k\,, \qquad i,j=1,2,3. \end{aligned} \tag{4}$$

here η is a Plank constant and the constant P is given by

$$\begin{aligned} P\ &=-1 \qquad \mathfrak{g}=o(4)\,,\\ &=0 \qquad \mathfrak{g}=e(3)\,,\\ &=1 \qquad \mathfrak{g}=o(3,1)\,, \end{aligned} \tag{5}$$

The Casimir operators

$$a^2 = g_k g_k - P l_k l_k \,, \qquad l = g_k l_k \,, \tag{6}$$

are supposed to be fixed. The Hamiltonian for the Kowalewski top is

$$h = l_1^2 + l_2^2 + 2l_3^2 - i(\alpha_1 g_+ + \alpha_2 g_-) \,, \qquad \alpha_k \in \mathbb{C} \,. \tag{7}$$

Here we use the general form for the Hamiltonian introduced in classical mechanics [4, 3]. Following Kowalewski we introduce new variables

$$\begin{aligned} q^- &= l_-^2 + 2i\alpha_1 g_- - \alpha_1^2 P \,, \\ q^+ &= l_+^2 + 2i\alpha_2 g_+ - \alpha_2^2 P \,, \end{aligned} \tag{8}$$

with $l_\pm = l_1 \pm i l_2$ and $g_\pm = g_1 \pm i g_2$. In classical mechanics these variables have been introduced in algebro-geometric approach for linearization of the Kowalewski flow [4].

Proposition 1 *In quantum mechanics operators $q^\pm$ are supercharges of SUSY algebra (1) for the following isospectral Hamiltonians*

$$h^{(1)} = h + 2\eta l_3 \,, \qquad h^{(2)} = h - 2\eta l_3 \,, \tag{9}$$

where h is the Hamiltonian of the Kowalewski top (7).

It can be proved by using the commutator relations introduced in [5].

For the Kowalewski top one finds that the central charge R is a diagonal operator with the following components

$$R_1 = q^+ q^- \,, \qquad R_2 = q^- q^+ \,, \qquad [R_j, h^{(j)}] = 0 \,, \tag{10}$$

which are the symmetry operators of the Hamiltonians $h^{(1)}$ and $h^{(2)}$, respectively. From the SUSY algebra (2) we get the pair of commuting operators

$$h = \frac{1}{2}(\, h^{(1)} + h^{(2)}\,) \,, \qquad k = \frac{1}{2}\{q_+, q_-\} + 2\eta^2 \{l_+, l_-\} \,, \tag{11}$$

which are Hamiltonian and second integral of motion for the Kowalewski top.

Now we shall construct a new two-dimensional integrable system by using the known SUSY algebra for the Kowalewski top. The SUSY algebra (1) can be rewritten as

$$[q^\pm, (\, h^{(1)} + h^{(2)}\,)\,] = \pm\{\,(h^{(1)} - h^{(2)}\,), q^\pm\,\} \tag{12}$$

or, by using the concise notations, as

$$[q^\pm, h] = \pm\{\, f, q^\pm\,\} \,, \qquad h = h^{(1)} + h^{(2)} \,, \quad f = h^{(1)} - h^{(2)} \,. \tag{13}$$

Proposition 2 *Let we have the SUSY algebra (2) for certain two-dimensional integrable system defined by four operators h, f and $q^{\pm}$. If the following equation in Δh can be solved*

$$[\Delta h, [q^+, q^-]\,] = [f, \{q^+, q^-\}\,]\,, \tag{14}$$

than the pair of the mutually commuting operators

$$\begin{aligned} \tilde{h} &= h + \Delta h = h^{(1)} + h^{(2)} + \Delta h\,, \\ \tilde{k} &= [q^+, q^-]\,, \qquad [\tilde{h}, \tilde{k}] = 0\,, \end{aligned} \tag{15}$$

defines a new two-dimensional integrable system.

It can be simply proved by using the Jacobi indentity for commutator relations. The new equation (14) can be rewritten as

$$\begin{aligned} [\Delta h, [q^+, q^-]\,] &= \frac{1}{2}[f, R_1 + R_2] = \frac{1}{2}\,\mathrm{tr}\,[\,(\sigma_3 H) \otimes R - R \otimes (\sigma_3 H)] \\ &= -\frac{1}{2}\,\mathrm{tr}\,[\,H \otimes (\sigma_2 R) - (\sigma_3 R) \otimes H] \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

where we used a tensor product of 2×2 matrices H, R (2-3) and σ_3 is a Pauli matrix.

The Kowalewski top with the integrals of motion (h, k) (11) and with the SUSY algebra defined by (1) has the integrable partner with integrals of motion $(\tilde{h}, \tilde{k})$ given by (15) in the one-parameter subset of orbits $\mathcal{O}$ ($a^2 = const$, $l = l_k g_k = 0$ (6)) of the Lie algebra $\mathfrak{g} = e(3)$. It is the Goryachev-Chapligin top with the following integrals of motion

$$\begin{aligned} \tilde{h} &= l_1^2 + l_2^2 + 4l_3^2 - 2i(\alpha_1 g_+ + \alpha_2 g_-)\,, \\ \tilde{k} &= -8i\left(l_3(l_1^2 + l_2^2 + 1/4) + i\{g_3, (\alpha_1 l_+ + \alpha_2 l_-)\}\right)\,, \\ \text{where} \qquad \Delta h &= 2l_3^2 - i(\alpha_1 g_+ + \alpha_2 g_-)\,, \end{aligned} \tag{16}$$

which is the partial solution of (14) on the corresponding orbit of $e(3)$. The another relations of these systems in framework of inverse scattering method are discussed in [2, 7].

Another example of the application of the Proposition 2 to the two-dimensional systems can be found in [6].

By definition of the Hamiltonians $h^{(j)}$ and of the supercharges $q^{\pm}$ the generator l_3 is distinguished generator and it can be changed to $l_3+\gamma$ without of violation of the superalgebra (2), here parameter γ being arbitrary. Therefore we can simple carry over all results on the systems with the shifted generator l_3. The corresponding systems are the Kowalewski gyrostat and Goryachev-Chapligin gyrostat [3, 5].

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